Int. J. Solids Structures, 1970, Vol. 6, pp. 1483 to 1498. Pergamon Press. Printed in Great Britain

INTERIOR MOTION OF AN ELASTIC HALF-SPACE DUE TO A NORMAL FINITE MOVING LINE LOAD ON ITS SURFACE†

F. R. NORWOOD

Sandia Laboratories, Albuquerque, New Mexico 87115

Abstract—The response of an elastic half-space to a normal impulsive semi-infinite line load moving parallel to its initial position is considered. The solution is found to be a superposition of cylindrical, spherical and conical waves, and is found by Cagniard's technique and by extending the real transformation of de Hoop to double Fourier integrals with singularities on the real axis of the transform variables. Velocities in the interior of the half-space are given for arbitrary values of Poisson's ratio in terms of single integrals and algebraic expressions. The case of a stationary load acting over one-quadrant of the bounding surface is obtained as the infinite velocity limit of the parallel motion. By a simple superposition the solution is obtained for a half-space acted upon by a finite line load or loaded on a finite rectangular region.

INTRODUCTION

THE present paper uses a recently developed technique to obtain solutions for an elastic half-space acted upon by line loads with finite characteristic dimensions [1]. The motivation for the effort is the fact that finite loads represent physical situations better than infinite loads; also, the motivation arises from the intrinsic value for the applied mechanician of a simple modification of well known techniques which allows him to solve heretofore unsolved problems. The solution is found by Cagniard's technique and by extending the real transformation of de Hoop to double Fourier integrals with singularities on the real axis of the transform variables. The technique has been carefully illustrated in solving the case of a stationary rectangular load and will now be applied to the more complex problem of a moving semi-infinite line load. The steps involved for the moving line load will be clearly delineated, and then a simple limiting process will be indicated for obtaining the response to a stationary load acting over one-quadrant of the bounding surface. The limiting solution agrees with the results of [1].

Previous work on transient solutions has largely been confined to infinite line loads [2–5], and there appears to be no work on finite or semi-infinite line loads. Thus, the present paper represents the first attempt in solving a different variety of line problems. However, it will be shown in the sequel that the solution for the corresponding infinite line load problem is contained in the solution for the finite line load. The solution is obtained by using double Fourier transforms on space, Laplace transforms on time and Cagniard's technique. The same theorems used in [6] can be employed here to rigorously justify the various steps in the inversion.

Gakenheimer [7] has found that the interior solution for an expanding ring load can be used for numerical calculations of the disturbance near the surface. The singularities characteristic of the surface appear in the interior solution as one approaches the surface,

†This work was supported by the U.S. Atomic Energy Commission.

e.g. Rayleigh waves. In view of these remarks, in the present work interior solutions are given. The results hold for arbitrary values of Poisson's ratio.

STATEMENT OF THE PROBLEM

In a rectangular coordinate system, consider the half-space $z > 0$, with bounding surface $z = 0$. The geometry of a semi-infinite line load moving parallel to its initial position is depicted in Fig. 1, where the x coordinate of the load at time t is given by ct, $c_1 < c < \infty$.†

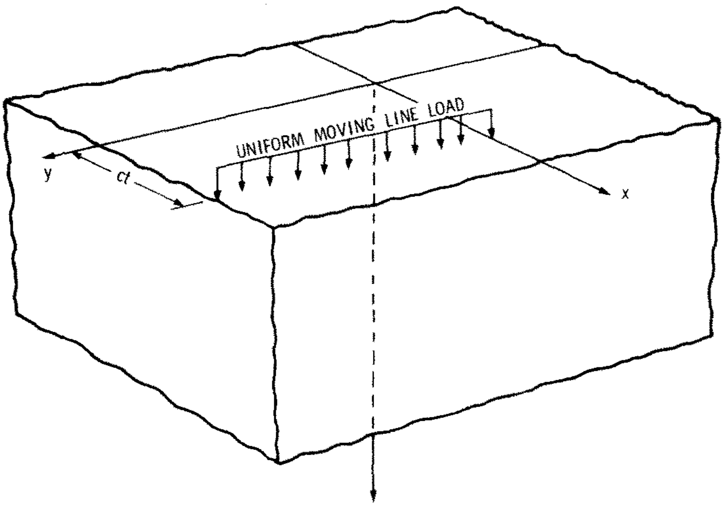

FIG. 1. Problem of a parallel moving surface line load.

The governing wave equations are

$$c_1^2\nabla^2\Phi = \frac{\partial^2\Phi}{\partial t^2}, \qquad c_2^2\nabla^2\mathbf{\Psi} = \frac{\partial^2\mathbf{\Psi}}{\partial t^2}, \qquad \nabla \cdot \mathbf{\Psi} = 0 \tag{1}$$

and the potentials Φ and $\mathbf{\Psi}$ are related to the displacements through $\mathbf{U} = \nabla\Phi + \nabla\times\mathbf{\Psi}$; where ∇^2 is the Laplacian operator, c_1 and c_2 are the wave speeds $vc_1^2 = \lambda+2\mu$, $vc_2^2 = \mu$, λ and μ are the Lamé constants and v is the material density. The stress–strain relations needed in the sequel are

$$\tau_{ij} = \lambda\nabla^2\Phi\delta_{ij}+2\mu\varepsilon_{ij}, \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{2}$$

where δ_{ij} is the Kronecker delta. The boundary conditions (at $z = 0$) for the problem are

$$\tau_{yz}(x, y, 0, t) = \tau_{xz}(x, y, 0, t) = 0 \qquad \tau_{zz}(x, y, 0, t) = -\delta(x-ct)H(y)H(x) \tag{3}$$

† The cases $0 \leq c < c_2$ and $c_2 < c < c_1$ can also be solved by the present technique (see [6] for more details).

where $\delta(t)$ is the Dirac delta function and $H(y)$ is the Heaviside unit function. The potentials Φ and Ψ (and hence the displacements and stresses) are required to vanish as $z \to \infty$; that is

$$\lim_{z\to\infty} (\Phi, \Psi, \mathbf{U}, \text{etc.}) = 0 \tag{4}$$

The initial conditions are taken as

$$\Phi(x, y, z, 0) = \Psi(x, y, x, 0) = \partial\Phi(x, y, z, 0)/\partial t = \partial\Psi(x, y, z, 0)/\partial t = 0 \tag{5}$$

representing quiescence at $t = 0$.

TRANSFORM SOLUTIONS

The Laplace and double Fourier transforms to be used here are defined, respectively, by the equations

$$\bar{f}(x, y, z, p) = \int_0^\infty f(x, y, z, t)\mathrm{e}^{-pt}\,\mathrm{d}t \tag{6a}$$

$$f(x, y, z, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(x, y, z, p)\mathrm{e}^{pt}\,\mathrm{d}p \tag{6b}$$

and

$$f^{FF}(k, v, z, t) = \iint_{-\infty}^{\infty} f(x, y, z, t)\mathrm{e}^{-i(kx+vy)}\,\mathrm{d}x\,\mathrm{d}y \tag{7a}$$

$$f(x, y, z, t) = \frac{1}{(2\pi)^2}\iint_{-\infty-i\varepsilon}^{\infty-i\varepsilon} f^{FF}(k, v, z, t)\mathrm{e}^{i(kx+vy)}\,\mathrm{d}k\,\mathrm{d}v \tag{7b}$$

where c is chosen to the right of any singularity of $\bar{f}$, and $-i\varepsilon$ lies within the strip of convergence [8].

Following the scheme of Ref. [1], the load given by the last of equations (3) is considered as the limit $\xi \to 0$ of the expression $[-\exp(-\xi y)\delta(x-ct)H(y)H(x)]$. The formal application of the transforms to this bracketted expression gives

$$l^*(k, v, c, \xi) = -(p+ikc)^{-1}(\xi+iv)^{-1}, \qquad -\infty < \operatorname{Im} v < \xi, \qquad -\infty < \operatorname{Im} k < p/c$$

where p is assumed to be a real positive quantity. Since $\xi > 0$, one may set ε equal to zero in equation (7b) and take advantage of the real transformation introduced by de Hoop [9] for problems in acoustics.† Therefore k and v may be assumed to be real quantities. After some algebra one finds that the Laplace transform of the velocity is given by

$$\bar{v}_j(x, y, z, p) = \bar{v}_{jd}(x, y, z, p) + \bar{v}_{js}(x, y, z, p), \qquad (j = x, y, z) \tag{8}$$

† The limit $\xi \to 0$ will be taken at an appropriate step in the solution.

where

$$\tilde{v}_{j\alpha}(x, y, z, p) = \frac{1}{(2\pi)^2\mu}\iint_{-\infty}^{\infty} F_{j\alpha}(k, v, p)I^*(k, v, c\xi)e^{-\eta_\alpha(k,v,p)z + i(kx+vy)p}\, dk\, dv. \qquad (\alpha = d, s) \quad (9)$$

$$F_{zd}(k, v, p) = -\eta_d(k, v, p)[k_s^2 + 2(k^2 + v^2)]T(k, v, p) \quad (10)$$

$$F_{zs}(k, v, p) = 2(k^2 + v^2)\eta_d(k, v, p)T(k, v, p) \quad (11)$$

$$F_{xd}(k, v, p) = ik[k_s^2 + 2(k^2 + v^2)]T(k, v, p) \quad (12)$$

$$F_{xs}(k, v, p) = -2ik\eta_s(k, v, p)\eta_d(k, v, p)T(k, v, p) \quad (13)$$

$$F_{yd}(k, v, p) = F_{xd}(v, k, p) = iv[k_s^2 + 2(k^2 + v^2)]T(v, k, p) \quad (14)$$

$$F_{ys}(k, v, p) = F_{xs}(v, k, p) = -2iv\eta_s(v, k, p)\eta_d(v, k, p)T(v, k, p) \quad (15)$$

$$T(k, v, p) = \{[k_s^2 + 2(k^2 + v^2)]^2 - 4(k^2 + v^2)\eta_s(k, v, p)\eta_d(k, v, p)\}^{-1} \quad (16)$$

$$\eta_d(k, v, p) = (k^2 + v^2 + k_d^2)^{\frac{1}{2}}, \qquad k_d = pa_1, \qquad c_1 a_1 = 1 \quad (17)$$

$$\eta_s(k, v, p) = (k^2 + v^2 + k_s^2)^{\frac{1}{2}}, \qquad k_s = pa_2, \qquad c_2 a_2 = 1 \quad (18)$$

with the requirement $\mathrm{Re}\,\eta_\alpha \geq 0$. In accordance with Lerch's theorem, it is sufficient to assume in these expressions that p is a real, positive number [10, p. 345], for this guarantees a unique inverse.

The real transformation introduced by de Hoop [9] for problems in acoustics, and later used by Mitra [11], Gakenheimer [6] and the author [1] for elastic half-space problems, will now be applied to the real variables k and v in equation (9). For the present case, this transformation may be written as $rk = p\omega x - pqy$, $rv = p\omega y + pqx$, $r^2 = x^2 + y^2$, and leads to the expression

$$\tilde{v}_{j\alpha}(x, y, z, p) = \frac{1}{(2\pi)^2\mu}\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} H_{j\alpha}(\omega, q, 1)I(\omega, q, a, \xi r/p)e^{-p[\eta_\alpha(\omega,q,1)z - i\omega r]}\, d\omega \quad (19)$$

where

$$\omega r H_{x\alpha}(\omega, q, 1) = (\omega x - qy)F_{x\alpha}(\omega, q, 1) \quad (20)$$

$$qrH_{y\alpha}(\omega, q, 1) = (\omega y + qx)F_{y\alpha}(\omega, q, 1) \quad (21)$$

$$H_{z\alpha}(\omega, q, 1) = F_{z\alpha}(\omega, q, 1) \quad (22)$$

$$I(\omega, q, a, \xi r/p) = -ar^2(i\omega x - iqy + ar)^{-1}(i\omega y + iqx + \xi r/p)^{-1} \quad (23)$$

TRANSFORM INVERSION

As was done in Ref. [1], the inversion of the Laplace transforms given will now be performed by Cagniard's technique. The steps involved will be clearly delineated.

In equation (19), let $\omega = i\sigma$ and consider σ as a complex variable. One easily deduces that, in addition to the singularities depicted in Fig. 2, there is a pole at $\sigma_1(q)$, $x\sigma_1(q) = ar - iqy$, which lies to the right of the imaginary axis. Recalling that, for $\alpha = d$, the intercept of the Cagniard Contour I of Fig. 2 is greater than or equal to $a_1 rR^{-1}$, one concludes that if

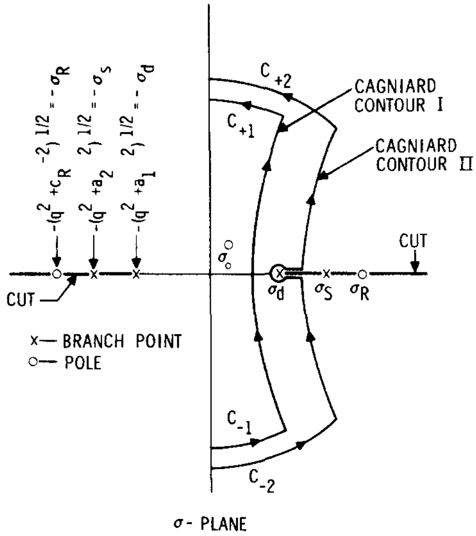

FIG. 2. Integration paths in the σ-plane.

$arx^{-1} > ra_1R^{-1}$, the pole $\sigma_1(q)$ will intercept the Cagniard path for

$$q^2 = q_{0d}^2 = z^2(R^2a^2 - a_1^2x^2)(rn)^{-2}, \qquad n^2 = y^2 + z^2 \tag{24}$$

To account for this when $\alpha = d$ the space is separated into the regions $ax^{-1} > a_1R^{-1}$ and $ax^{-1} < a_1R^{-1}$. As indicated by Gakenheimer [6], the inversion is more complicated for $\alpha = s$ than for $\alpha = d$. Indeed, when $\alpha = s$ the space is separated into the regions $arx^{-1} < a_2rR^{-1} < a_1$, $arx^{-1} < a_1 < a_2rR^{-1}$, $a_1 < arx^{-1} < a_2rR^{-1}$ and $a_1 < a_2rR^{-1} < arx^{-1}$, where x positive is required only in the last two regions. The inversion will now be performed for each region.

Region I

$\alpha = d, ax^{-1} < a_1R^{-1}$. In this region $\sigma_1(q)$ lies within the closed contour Γ_d for all values of q and $x > 0$, $\Gamma_d = \text{Im}\sigma \text{ axis} + C_{-1} + \text{Cagniard contour I} + C_{+1}$, where the various components are shown in Fig. 2. Therefore, by residue theory, one immediately obtains

$$\begin{aligned}\tilde{v}_{jd}(x, y, z, p) = \frac{1}{2\pi\mu}\Big\{&H(x)\int_{-\infty}^{\infty} H_{jd}(i\sigma_1(q), q, 1)w(q, \xi, a)e^{-p[\eta_d(i\sigma_1(q), q, 1)z + \sigma_1(q)r]}\,dq \\ &- H(y)\int_{-\infty}^{\infty} H_{jd}(i\sigma_0(q), q, 1)w(q, \xi, a)e^{-p[\eta_d(i\sigma_0(q), q, 1)z + \sigma_0(q)r]}\,dq\Big\} \\ &+ \frac{1}{(2\pi)^2\mu i}\int_{-\infty}^{\infty} dq\left(\int_{\Gamma_1} H_{jd}(i\sigma, q, 1)l(i\sigma, q, a, \xi r/p)e^{-p[\eta_d(i\sigma, q, 1)z + \sigma r]}\,d\sigma\right)\end{aligned} \tag{25}$$

where Γ_1 is the Cagniard contour I of Fig. 2, and

$$w(q, \xi, a) = -ar[iqr - (ay - \xi r/p)]^{-1}.$$

1. To find the contribution from the first term of equation (25), take the limit $\xi \to 0$ and introduce the change of variable $qr = i(sx-ay)$, where s is a complex variable. Then $\sigma_1(q)$ becomes $\sigma_1(s)$, where $r\sigma_1(s) = ax+sy$, and the integral may be written as†

$$\bar{I}_{1j}(x, y, z, p) = \frac{aH(x)}{\mu 2\pi i}\int_{ay/x-i\infty}^{ay/x+i\infty} F_{jd}(ia, is, 1)e^{-p[\eta_d(ia,is,1)z+ax+sy]}\,\mathrm{d}s/s \tag{26}$$

The singularities of $F_{jd}(ia, is, 1)$ are branch points at $s = \pm(a_1^2-a^2)^{\frac{1}{2}}$ and $s = \pm(a_2^2-a^2)^{\frac{1}{2}}$, and simple poles at $s = \pm(c_R^{-2}-a^2)^{\frac{1}{2}}$. If y is positive, then, since $xc > Rc_1 \geq rc_1$ implies $x^2c^2 > (x^2+y^2)c_1^2$ and $ya < x(a_1^2-a^2)^{\frac{1}{2}}$, the contour in equation (26) lies between the pole at $s = 0$ and the branch point $s = (a_1^2-a^2)^{\frac{1}{2}}$. Consequently, this contour may be replaced by a contour along the imaginary axis and indented such that the origin lies to the left of the contour. If y is negative, then the contour of equation (26) may be replaced by a contour along the imaginary axis and indented such that the origin lies to the right of the contour. In view of these remarks Fig. 3 may be used in this case *mutatis mutandis* and the evaluation proceeds as in Ref. [1]. The final expression for $y > 0$ may be written as

$$I_{1j}(x, y, z, t) = H(x)H[t-ax-n(a_1^2-a^2)^{\frac{1}{2}}]B_{jd}(x, y, z, t-ax) \tag{27}$$

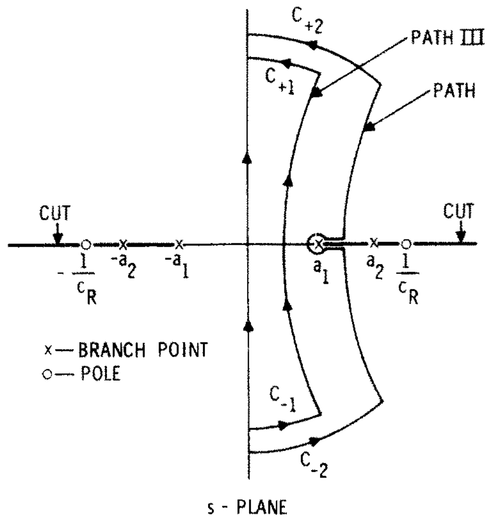

FIG. 3. Integration paths in the s-plane.

where

$$\pi\mu c B_{jd}(x, y, z, t) = \mathrm{Re}[t^2-(a_1^2-a^2)n^2]^{-\frac{1}{2}}F_{jd}(ia, is_4, 1)\eta_d(ia, is_4, 1) \tag{28}$$

$$s_4(y, z, t) = \frac{t|y|}{n^2}+\frac{iz}{n^2}[t^2-(a_1^2-a^2)n^2]^{\frac{1}{2}} \tag{29}$$

and $|y|$ is the absolute value of y. Using this notation, the contribution for negative values of y is given by equation (27), with $B_{jd}(x, y, z, t)$ now defined by

$$B_{jd}(x, y, z, t) = \pm B_{jd}(x, -y, z, t) \tag{30}$$

† The roman numeral identifies the region, and the subscript 1 the contribution from the pole at $\sigma_1(q)$.

where the positive sign applies only for $j = y$. $t = t_{dc} = ax + n(a_1^2 - a^2)^{\frac{1}{2}}$ in equation (27) represents the arrival time of a conical dilatational wave which trails behind the point $x = ct$, $y = z = 0$.

2. To evaluate the contribution from the second term of equation (25), one takes the limit $\xi \to 0$ and introduces the change of variable $iqr = ys$; s is a complex variable. Then $\sigma_0(q)$ becomes $\sigma_0(s)$, where $r\sigma_0(s) = xs$, and the integral is transformed into (no sum on j)†

$$\bar{\mathrm{I}}_{2j}(x, y, z, p) = (1-\delta_{jy})\frac{aH(y)}{\mu 2\pi i}\int_{-i\infty}^{i\infty} (s-a)^{-1}F_{jd}(is, 0, 1)\mathrm{e}^{-p[\eta_d(is,0,1)z+xs]}\,\mathrm{d}s \qquad (31)$$

By omitting $H(y)$ in equation (31), one obtains the Laplace transform of the dilatational contribution for an infinite line load moving with velocity c.

The singularities of the integrand of (31) are a pole at $s = a$ plus the singularities shown on Fig. 3 and for $x < 0$ the contour is closed to the right of the imaginary axis. Since in this region $a_1 x > aR \geq an$, then $a < a_1 x n^{-1}$. But $a_1 x n^{-1}$ is the intercept of the Cagniard path, and therefore the pole at $s = a$ contributes for $x > 0$, $ax^{-1} < a_1 R^{-1}$. Hence the contribution for $x > 0$ is given by

$$\begin{aligned}\mathrm{I}_{2j}(x, y, z, t) &= (1-\delta_{jy})H(y)H(t-a_1\rho)\mathrm{Re}(s_1-a)^{-1}aK_{j1}(x, z, t)\\ &\quad -(1-\delta_{jy})a\mu^{-1}H(y)F_{jd}(ia, 0, 1)\delta[t-ax-(a_1^2-a^2)^{\frac{1}{2}}z] \qquad (32)\\ \pi\mu K_{j1} &= (t^2-a_1^2\rho^2)^{-\frac{1}{2}}F_{jd}(is_1, 0, 1)\eta_d(is_1, 0, 1),\\ s_1(x, z, t) &= \frac{t|x|}{\rho^2}+\frac{iz}{\rho^2}(t^2-a_1^2\rho^2)^{\frac{1}{2}},\end{aligned}$$

$\rho^2 = x^2+z^2$, and $|x|$ is the absolute value of x. The equation $t = ax + z(a_1^2 - a^2)^{\frac{1}{2}}$ represents a plane tangent to the cone $t = ax + n(a_1^2 - a^2)^{\frac{1}{2}}$ and intersecting the surface of the half-space along the line $x = ct$.

For $x < 0$ the contour in (31) is closed to the left of the imaginary axis, and the contribution is found to be $\mathrm{I}_{2y}(x, y, z, t) = 0$,

$$\mathrm{I}_{2x}(x, y, z, t) = H(y)H(t-a_1\rho)\mathrm{Re}(s_1+a)^{-1}aK_{x1}(x, z, t) \qquad (33)$$

$$\mathrm{I}_{2z}(x, y, z, t) = -H(y)H(t-a_1\rho)\mathrm{Re}(s_1+a)^{-1}aK_{z1}(x, z, t) \qquad (34)$$

3. The evaluation of the last term of equation (25) proceeds in precisely the same way as for the last term of equation (42), Ref. [1]. It may readily be seen that the total contribution is‡

$$v_{jd}(x, y, z, t) = \mathrm{I}_{1j}(x, y, z, t)+\mathrm{I}_{2j}(x, y, z, t)+\mathrm{I}_{3j}(x, y, z, t) \qquad (35)$$

in which

$$\mathrm{I}_{3j}(x, y, z, t) = H(t-t_d)\int_0^{q_1(t)} \mathscr{H}_{jd}(t, \sigma_5, q, a)\,\mathrm{d}q \qquad (36)$$

$$\mathscr{H}_{jd}(t, \sigma_5, q, a) = \mathrm{Re}(t^2-t_{qd}^2)^{-\frac{1}{2}}\mathscr{N}_{jd}(\sigma_5, q, a)\eta_s(i\sigma_5, q, 1) \qquad (37)$$

$$\mathscr{N}_{j\alpha}(\sigma, q, a) = A_j(\sigma, q, a)D_{j\alpha}(\sigma, q, a), \quad \text{(no sum on } j) \qquad (38)$$

$$\sigma A_x(\sigma, q, a) = -ar[\sigma^2 xyg(\sigma, a)+q^2y(\sigma y^2+axr)] \qquad (39)$$

† The subscript 2 identifies the contribution from the pole at $\sigma_0(q)$.
‡ The subscript 3 identifies the contribution from the required Cagniard path.

$$qA_y(\sigma, q, a) = -iar[\sigma^2 y^2 g(\sigma, a) + q^2 x(\sigma x^2 - axr)] \tag{40}$$

$$A_z(\sigma, q, a) = -ar^2\sigma y g(\sigma, a) - ar^2 q^2 xy \tag{41}$$

$$F_{j\alpha}(i\sigma, q, 1) = \pi^2\mu[g^2(\sigma, a) + q^2y^2][\sigma^2y^2 + q^2x^2]D_{j\alpha}(\sigma, q, a) \tag{42}$$

$$g(\sigma, a) = \sigma x - ar \tag{43}$$

$$t_d = a_1 R, q_1(t) = R^{-1}(t^2 - a_1^2 R^2)^{\frac{1}{2}} \tag{44}$$

$$t_{qd} = R(a_1^2 + q^2)^{\frac{1}{2}}, \qquad R^2 = x^2 + y^2 + z^2 \tag{45}$$

$$\sigma_5(x, y, z, t) = \frac{tr}{R^2} + \frac{iz}{R^2}(t^2 - t_{qd}^2)^{\frac{1}{2}} \tag{46}$$

Region II

$\sigma = d, ax^{-1} > a_1 R^{-1}, x > 0$. In this region $\sigma_1(q)$ lies within Γ_d only for $|q|$ greater than q_{0d}. Furthermore, when $q^2 = q_{0d}^2$ and $t = t_L = R^2 ax^{-1}$, the pole $\sigma_1(q)$ lies on the Cagniard contour I. Changes must therefore be effected in the first and third terms of equation (25) to find $v_{jd}(x, y, z, t)$ in this region. Notice that t_L may also be written as†

$$t_L = R^2 ax^{-1} = t_d R c_1 a x^{-1} \tag{47}$$

and hence the inequality $ax^{-1} \lesseqgtr a_1 R^{-1}$ may also be written as $t_L \lesseqgtr t_d$.

1. The contribution from the pole at $\sigma_1(q)$ is obtained from the first term of equation (25) by restricting q^2 to values greater than q_{0d}^2. In the resulting expression one now takes the limit $\xi \to 0$ and introduces again the change of variable $qr = i(sx - ay)$ to obtain equation (26), where now the path of integration consists of the line segments L_1 and L_2 defined by $L_1 : xs = ay - \eta$, $L_2 : xs = ay + \eta$, $q_{0d}r < x\eta < \infty$ as shown in Fig. 4. The integration by Cagniard's technique proceeds as for equation (26), and the resulting expression may be written as

$$\mathrm{II}_{1j}(x, y, z, t) = H(x)H(t - t_L)B_{jd}(x, y, z, t - ax) \tag{48}$$

where B_{jd} has been defined for equation (27).

2. The contribution from the pole at $\sigma_0(q)$ is equal to the second term of equation (25), and therefore equal to equation (31). However, in this region the pole at $s = a$ lies to the right of the Cagniard contour and does not contribute. Thus, the contribution for $x > 0$ is given by

$$\mathrm{II}_{2j}(x, y, z, t) = (1 - \delta_{jy})H(y)H(t - a_1\rho)\mathrm{Re}(s_1 - a)^{-1} a K_{j1}(x, z, t) \tag{49}$$

For $x < 0$, the contribution is given by

$$\mathrm{II}_{2j}(x, y, z, t) = \mathrm{I}_{2j}(x, y, z, t) \tag{50}$$

and $\mathrm{I}_{2j}(x, y, z, t)$ is defined by equations (33) and (34) for $j = x, z$, and $\mathrm{II}_{2y}(x, y, z, t) = \mathrm{I}_{2y}(x, y, z, t) = 0$.

3. The contribution from the Cagniard contour I may be considered as the limit $\varepsilon \to 0$ of

$$\frac{1}{(2\pi)^2\mu i}\left(\int_{-\infty}^{-q_{0d}-\varepsilon} + \int_{-q_{0d}+\varepsilon}^{q_{0d}-\varepsilon} + \int_{q_{0d}+\varepsilon}^{\infty}\right)F(q)\,\mathrm{d}q \tag{51}$$

† $t = t_L$ represents a hemispherical surface of radius $ct/2$ centered about the point $x = ct/2$, $y = z = 0$. The same surface was found in [6].

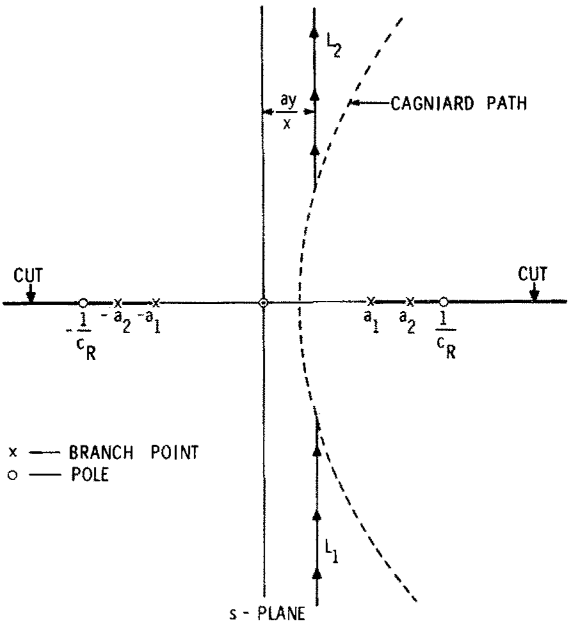

FIG. 4. Auxiliary integration path for Region II.

where $F(q)$ is a generic expression for the q-integrand in the third term of equation (25).† Proceeding the same way as for the last term of equation (42), Ref. [1], and by appealing to Fig. 5, one immediately finds that (before taking the limit $\varepsilon \to 0$)

$$\begin{aligned} \mathrm{II}_{3j}(x, y, z, t) = {} & [H(t-t_d)-H(t-t_d^*+\varepsilon)] \int_0^{q_1(t)} \mathscr{H}_{jd}(t, \sigma_5, q, a)\,\mathrm{d}q \\ & + H(t-t_d^*+\varepsilon) \int_0^{q_{0d}-\varepsilon} \mathscr{H}_{jd}(t, \sigma_5, q, a)\,\mathrm{d}q \\ & + H(t-t_d^*-\varepsilon) \int_{q_{0d}+\varepsilon}^{q_1(t)} \mathscr{H}_{jd}(t, \sigma_5, q, a)\,\mathrm{d}q \end{aligned} \tag{52}$$

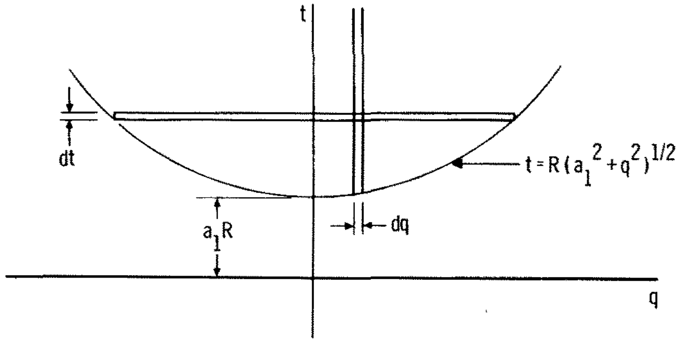

FIG. 5. Region of integration in the t–q plane.

† This is equivalent to the separation used in [2].

where $t_d^* = R(a_1^2+q_{0d}^2)^{\frac{1}{2}}$. Since the singularity exists only for $t = t_L$, by again considering Fig. 5 one can also write

$$\text{II}_{3j}(x, y, z, t) = [H(t-t_d)-H(t-t_L+\varepsilon)]\int_0^{q_1(t)} \mathscr{H}_{jd}(t, \sigma_5, q, a)\,\mathrm{d}q$$
$$+H(t-t_L-\varepsilon)\left[\int_0^{q_{0d}-\varepsilon}+\int_{q_{0d}+\varepsilon}^{q_1(t)}\right]\mathscr{H}_{jd}(t, \sigma_5, q, a)\,\mathrm{d}q \tag{53}$$

Proceeding even further, since the singularity is not there for $t = t_L+\varepsilon$, the last two integrals can be combined to yield

$$\text{II}_{3j}(x, y, z, t) = [H(t-t_d)-H(t-t_L+\varepsilon)+H(t-t_L-\varepsilon)]\int_0^{q_1(t)} \mathscr{H}_{jd}(t, \sigma_5, q, a)\,\mathrm{d}q \tag{54}$$

It follows that†

$$\text{II}_{3j}(x, y, z, t) = H(t-t_d)P(t_L, q_{0d})\int_0^{q_1(t)} \mathscr{H}_{jd}(t, \sigma_5, q, a)\,\mathrm{d}q \tag{55}$$

where the symbol $P(t_L, q_{0d})$ before the integral specifies that, when $t = t_L$, the value of the integral is the *Cauchy principal value* [12]. In other words, when $t = t_L$, equation (55), or equivalently equation (52), gives

$$\text{II}_{3j}(x, y, z, t_L) = \lim_{\varepsilon\to 0}\left(\int_0^{q_{0d}-\varepsilon}+\int_{q_{0d}+\varepsilon}^{q_1(t)}\right)\mathscr{H}_{jd}(t_L, \sigma_5, q, a)\,\mathrm{d}q \tag{56}$$

Thus, in this region the total contribution is given by

$$v_{jd}(x, y, z, t) = \text{II}_{1j}(x, y, z, t)+\text{II}_{2j}(x, y, z, t)+\text{II}_{3j}(x, y, z, t) \tag{57}$$

where the $\text{II}_{ij}(x, y, z, t)$ are defined by equations (48)–(50) and (55).

As indicated by Gakenheimer [6], when $\alpha = s$ the inversion is more complicated than for the case $\alpha = d$ due to the appearance of head waves. However, these complications are similar to those which arose in Ref. [1] for $\alpha = s$ and in regions I and II.

Region III

$\alpha = s, ax^{-1} < a_2R^{-1}$ and $arx^{-1} < a_1$. In this region the pole $\sigma_1(q)$ lies to the left of the Cagniard contours of Fig. 2 for all values of q. Therefore, by residue theory, one immediately obtains

$$\tilde{v}_{js}(x, y, z, p) = \frac{1}{2\pi\mu}\left\{H(x)\int_{-\infty}^{\infty} H_{js}(i\sigma_1(q), q, 1)w(q, \xi, a)\mathrm{e}^{-p[\eta_s(i\sigma_1(q), q, 1)z+\sigma_1(q)r]}\,\mathrm{d}q\right.$$
$$\left.-H(y)\int_{-\infty}^{\infty} H_{js}(i\sigma_0(q), q, 1)w(q, \xi, a)\mathrm{e}^{-p[\eta_s(i\sigma_0(q), q, 1)z+\sigma_0(q)r]}\,\mathrm{d}q\right\}$$
$$+\frac{1}{(2\pi)^2\mu i}\int_{-\infty}^{\infty}\mathrm{d}q\left(\int_{\Gamma_s} H_{js}(i\sigma, q, 1)l(i\sigma, q, a, \xi r/p)\mathrm{e}^{-p[\eta_s(i\sigma, q, 1)z+\sigma r]}\,\mathrm{d}\sigma\right) \tag{58}$$

where Γ_s is either the Cagniard contour I or the Cagniard contour II of Fig. 2, depending on the ratio of r to z and the values of q.

† These results are in accordance with the findings of [6].

1. The contribution from each term of this equation is found in the same way as for the corresponding term of (25). Thus, in the first term one takes the limit $\xi \to 0$ and introduces the change of variable $qr = i(sx - ay)$ to obtain

$$\tilde{\text{III}}_{1j}(x, y, z, p) = \frac{aH(x)}{\mu 2\pi i} \int_{ay/x - i\infty}^{ay/x + i\infty} F_{js}(ia, is, 1) \mathrm{e}^{-p[\eta_s(ia, is, 1)z + ax + sy]} \, \mathrm{d}s/s \tag{59}$$

The right side differs from the right side of (26) only in the subscript s for $F_{j\alpha}$ and η_α, and it is easy to see that the singularities of (59) are the same as those of (26). Also, the discussion on contours for (26) is applicable here. For $y > 0$ the final expression may be written as

$$\text{III}_{1j}(x, y, z, t) = H(x)[H(t - t_{sdc}) - H(t - t_{sc})]B_{js}(x, y, z, t - ax)$$
$$+ H(x)H(t - t_{sc})G_{js}(x, y, z, t - ax) \tag{60}$$

where†

$$t_{sdc}(x, y, z) = z(a_2^2 - a_1^2)^{\frac{1}{2}} + y(a_1^2 - a^2)^{\frac{1}{2}} + ax \tag{61}$$

$$t_{sc}(x, y, z) = n(a_2^2 - a^2)^{\frac{1}{2}} + ax \tag{62}$$

$$\eta \mu B_{js}(x, y, z, t) = \text{Im}[(a_2^2 - a^2)n^2 - t^2]^{-\frac{1}{2}} a s_5^{-1} F_{js}(ia, is_5, 1)\eta_s(ia, is_5, 1) \tag{63}$$

$$s_5(y, z, t) = \frac{t|y|}{n^2} + \frac{z}{n^2}[(a_2^2 - a^2)n^2 - t^2]^{\frac{1}{2}} \tag{64}$$

$$\pi \mu G_{js}(x, y, z, t) = \text{Re}[t^2 - (a_2^2 - a^2)n^2]^{-\frac{1}{2}} a s_6^{-1} F_{js}(ia, is_6, 1)\eta_s(ia, is_6, 1) \tag{65}$$

$$s_6(y, z, t) = \frac{t|y|}{n^2} + \frac{iz}{n^2}[t^2 - (a_2^2 - a^2)n^2]^{\frac{1}{2}} \tag{66}$$

Using this notation, the contribution for negative values of y is given by equation (60), with B_{js} and G_{js} now defined by

$$B_{js}(x, y, z, t) = \pm B_{js}(x, -y, z, t) \tag{67}$$

$$G_{js}(x, y, z, t) = \pm G_{js}(x, -y, z, t) \tag{68}$$

where the positive sign applies only for $j = y$. $t = t_{sc}$ and $t = t_{sdc}$ in equation (60) represent, respectively, the arrival times of conical shear waves and head waves which trail behind the point $x = ct$, $y = z = 0$.

2. The contribution from the second term of (58) is found by taking the limit $\xi \to 0$ and again introducing the change of variable $iqr = ys$. This yields the expression (no sum on j)

$$\tilde{\text{III}}_{2j}(x, y, z, p) = (1 - \delta_{jy}) \frac{aH(y)}{\mu 2\pi i} \int_{-i\infty}^{i\infty} (s - a)^{-1} F_{js}(is, 0, 1) \mathrm{e}^{-p[\eta_s(is, 0, 1)z + xs]} \, \mathrm{d}s \tag{69}$$

As was the case for equation (31), by omitting $H(y)$ in (69), one obtains the Laplace transform of the shear contribution for an infinite line load moving with velocity c. The inversion of

† t_{sc} and t_{sdc} also appear in [6].

(69) for $x > 0$ gives

$$\begin{aligned}
\mathrm{III}_{2j}(x, y, z, t) &= (1-\delta_{jy})H(y)H(t-a_2\rho)\,\mathrm{Re}\,a(s_3-a)^{-1}M_{j3}(x, z, t) \\
&\quad +(1-\delta_{jy})H(y)\{H[t-a_1|x|-z(a_2^2-a_1^2)^{\frac{1}{2}}]-H(t-a_2\rho)\} \\
&\quad \times \mathrm{Im}\,a(s_2-a)^{-1}L_{j2}(x, z, t) \\
&\quad -(1-\delta_{jy})a\mu^{-1}H(y)H(x\rho^{-1}-ac_2)F_{js}(ia, 0, 1)\delta[t-ax-z(a_2^2-a^2)^{\frac{1}{2}}] \qquad (70) \\
\pi\mu L_{j2} &= (a_2^2\rho^2-t^2)^{-\frac{1}{2}}F_{js}(is_2, 0, 1)\eta_s(is_2, 0, 1) \\
s_2(x, z, t) &= \frac{t|x|}{\rho^2}-\frac{z}{\rho^2}(a_2^2\rho^2-t^2)^{-\frac{1}{2}}, \\
\pi\mu M_{j3} &= (t^2-a_2^2\rho^2)^{-\frac{1}{2}}F_{js}(is_3, 0, 1)\eta_s(is_3, 0, 1), \\
s_3(x, z, t) &= \frac{t|x|}{\rho^2}+\frac{iz}{\rho^2}(t^2-a_2^2\rho^2)^{\frac{1}{2}}
\end{aligned}$$

For $x < 0$ the contour is closed to the left of the imaginary axis, and therefore the pole at $s = a$ is not included. Thus, for $x < 0$ one finds that

$$\begin{aligned}
\mathrm{III}_{2x}(x, y, z, t) &= H(y)\{H[t-a_1|x|-z(a_2^2-a_1^2)^{\frac{1}{2}}]-H(t-a_2\rho)\}\,\mathrm{Im}\,a(s_2+a)^{-1}L_{x2}(x, z, t) \\
&\quad +H(y)H(t-a_2\rho)\,\mathrm{Re}\,a(s_3+a)^{-1}M_{x3}(x, z, t) \qquad (71)
\end{aligned}$$

$$\mathrm{III}_{2z}(x, y, z, t) = -H(y)\{H[t-a_1|x|-z(a_2^2-a_1^2)^{\frac{1}{2}}]-H(t-a_2\rho)\}\,\mathrm{Im}\,a(s_2+a)^{-1}L_{z2}(x, z, t) \qquad (72)$$

$$\mathrm{III}_{2y}(x, y, z, t) \equiv 0 \qquad (73)$$

3. The evaluation of the last term of (58) proceeds in precisely the same way as for the last term of equation (42), Ref. [1]. Thus, the total contribution is

$$v_{js}(x, y, z, t) = \mathrm{III}_{1j}(x, y, z, t)+\mathrm{III}_{2j}(x, y, z, t)+\mathrm{III}_{3j}(x, y, z, t) \qquad (74)$$

in which

$$\begin{aligned}
\mathrm{III}_{3j}(x, y, z, t) &= H(t-t_s)\int_0^{q_2(t)} \mathrm{Re}(t^2-t_{qs}^2)^{-\frac{1}{2}}\mathcal{N}_{js}(\sigma_6, q, a)n_6\,\mathrm{d}q \\
&\quad +H(rR^{-1}-c_2a_1)[H(t-t_H)-H(t-t_s)]\int_0^{q_3(t)} \mathrm{Im}(t_{qs}^2-t^2)^{-\frac{1}{2}}\mathcal{N}_{js}(\sigma_7, q, a)n_7\,\mathrm{d}q \\
&\quad +H(rR^{-1}-c_2a_1)[H(t-t_s)-H(t-t_B)]\int_{q_2(t)}^{q_3(t)} \mathrm{Im}(t_{qs}^2-t^2)^{-\frac{1}{2}}\mathcal{N}_{js}(\sigma_7, q, a)n_7\,\mathrm{d}q \qquad (75)
\end{aligned}$$

$\mathcal{N}_{j\alpha}$ is defined by (38)–(43), $n_j = \eta_s(i\sigma_j, q, 1)$ for $j = 6, 7$, and

$$t_s = a_2R, \qquad t_H = a_1r+z(a_2^2-a_1^2)^{\frac{1}{2}}, \qquad t_B = R^2(a_2^2-a_1^2)^{\frac{1}{2}}/z,$$

$$q_2(t) = R^{-1}(t^2-a_2^2R^2)^{\frac{1}{2}},$$

$$q_3(t) = \{r^{-2}[t-z(a_2^2-a_1^2)^{\frac{1}{2}}]^2-a_1^2\}^{\frac{1}{2}}$$

$$\sigma_6(x, y, z, t) = \frac{tr}{R^2}+\frac{iz}{R^2}(t^2-t_{qs}^2)^{\frac{1}{2}}, \qquad t_{qs} = R(a_2^2+q^2)^{\frac{1}{2}},$$

$$\sigma_7(x, y, z, t) = \frac{tr}{R^2}-\frac{z}{R^2}(t_{qs}^2-t^2)^{\frac{1}{2}}$$

Region IV

$\alpha = s, a_1 < arx^{-1} < a_2rR^{-1}, x > 0$. In this region the Cagniard contour II of Fig. 2 is required. The pole $\sigma_1(q)$ is included for all values of q different from zero. However, when $q = 0$, the pole intersects the contour at points on the horizontal straight lines to the right of σ_d. The contributions must therefore be considered as the limit $\varepsilon \to 0$ of the expressions to follow containing ε.

1. The contribution from the pole at $\sigma_1(q)$ is given by the first term of (58) by requiring $|q| \geq \varepsilon > 0$. As for Region III, the integral may be reduced to the right side of (59), where now $0 < \varepsilon \leq |\text{Im}\, s|$. The inequalities $a_1 < arx^{-1}$ and $arx^{-1} < a_2rR^{-1}$ imply $(a_1^2-a^2)^{\frac{1}{2}} < ayx^{-1}$ and $ax^{-1} < (a_2^2-a^2)^{\frac{1}{2}}n^{-1}$, respectively; that is, $(a_1^2-a^2)^{\frac{1}{2}} < ayx^{-1} < y(a_2^2-a^2)^{\frac{1}{2}}n^{-1}$. Hence, as $\varepsilon \to 0$, one obtains the contribution†

$$IV_{1j}(x, y, z, t) = H(x)[H(t-t_E)-H(t-t_{sc})]B_{js}(x, y, z, t-ax)+H(x)H(t-t_{sc})G_{js}(x, y, z, t-ax) \tag{76}$$

$$t_E(x, y, z) = \eta_s(ia, iayx^{-1}, 1)z+ax+ay^2x^{-1} = zx^{-1}(a_2^2x^2-a^2r^2)^{\frac{1}{2}}+ar^2x^{-1} \tag{77}$$

As in the corresponding contribution in Ref. [6], p. 51, the second term of (76) is again the conical, shear wave found in Region III, equation (60); whereas the first term has the algebraic form of the plane head wave of Region III, but it now lacks the plane wave front at $t = t_{sdc}$.

2. The contribution from the pole at $\sigma_0(q)$ is given by the second term of (60) with no additional constraints. Therefore, one immediately finds that in this region the contribution is given by

$$IV_{2j}(x, y, z, t) \equiv III_{2j}(x, y, z, t). \tag{78}$$

3. The contribution from the Cagniard path may be obtained by noting that when $q = 0$ the pole $\sigma_1(q)$ intersects the path at $t = t_E$, $t_H < t_E < t_s$. Proceeding as for Region II, one finds that the contribution may be written as

$$IV_{3j}(x, y, z, t) = III_{3j}(x, y, z, t), \tag{79}$$

provided that the second integral in (75) be preceded by the symbol $P(t_E, 0)$ defined for equation (55). With this proviso, the total contribution is

$$v_{js}(x, y, z, t) = IV_{1j}(x, y, z, t)+III_{2j}(x, y, z, t)+III_{3j}(x, y, z, t) \tag{80}$$

Region V

$\alpha = s, a_1 < a_2rR^{-1} < arx^{-1}, x > 0$. In this region the Cagniard contour II of Fig. 2 is also required, and the pole $\sigma_1(q)$ contributes only for $|q|$ greater than q_{0s}, where

$$q_{0s}^2 = z^2(R^2a^2-a_2^2x^2)(rn)^{-2} \tag{81}$$

Moreover, the pole lies on the Cagniard contour II for $q^2 = q_{0s}^2$ and $t = t_L$. Changes must therefore be effected in the first and third terms of (58) to find $v_{js}(x, y, z, t)$ in this region. Corresponding to (47), one can write

$$t_L = R^2ax^{-1} = t_sRc_2ax^{-1} \tag{82}$$

and hence the inequality $a_2rR^{-1} \lessgtr arx^{-1}$ may also be written as $t_s \lessgtr t_L$.

† Compare equations (60) and (76).

1. For the contribution from the pole at $\sigma_1(q)$, the discussion in the paragraph preceding equation (48) applies here, provided that q_{0d} is replaced by q_{0s}. The resulting expression in this case is given by

$$V_{1j}(x, y, z, t) = H(x)H(t-t_L)G_{js}(x, y, z, t-ax) \tag{83}$$

where G_{js} is defined by equation (65) for $y > 0$, and by (68) for $y < 0$.

2. The pole at $\sigma_0(q)$ again gives a contribution equal to the second term of (58). Therefore

$$V_{2j}(x, y, z, t) = \mathrm{III}_{2j}(x, y, z, t) \tag{84}$$

3. The contribution from the Cagniard contour may be easily deduced by noting that $t_L > t_s$, and hence the pole intersects the path on the branch of the hyperbola in Fig. 2. The evaluation proceeds as for Region II, and one finds that

$$v_{js}(x, y, z, t) = V_{1j}(x, y, z, t)+\mathrm{III}_{2j}(x, y, z, t)+V_{3j}(x, y, z, t) \tag{85}$$

and

$$V_{3j}(x, y, z, t) = \mathrm{III}_{3j}(x, y, z, t) \tag{86}$$

provided that the first term of equation (75) be interpreted as a Cauchy principal value for $t = t_L$ and $q = q_{0s}$; that is, provided that

$$H(t-t_s)P(t_L, q_{0s})\int_0^{q_2(t)} \mathrm{Re}(t^2-t_{qs}^2)^{-\frac{1}{2}}\mathcal{N}_{js}(\sigma_6, s)n_6\,\mathrm{d}q \tag{87}$$

be used in place of the first term of (75).

The surfaces $t = t_s$, $t = t_B$, etc., have been discussed in Refs. [1 and 6], where the appropriate figures appear. With the exception of the surfaces $t = t_B$ and $t = t_L$ all of the other surfaces can be used to generate the wave pattern for y positive. For example, by continuing the curve $t = t_d$ parallel to the y-axis, the surface $t = a_1\rho$ in equation (32) is generated.

The solution for a uniform moving line load, acting on the interval $x = ct$, $-b < y < b$, $z = 0$, may immediately be written as

$$v_j^{FL}(x, y, z, t) = v_j(x, y+b, z, t)-v_j(x, y-b, z, t) \tag{88}$$

where the superscript FL stands for finite length, and, in accordance with equation (8),

$$v_j(x, y, z, t) = v_{jd}(x, y, z, t)+v_{js}(x, y, z, t) \tag{89}$$

LOAD ON ONE QUADRANT OF THE SURFACE

The solution for this case may be obtained by noting the relation

$$\delta(f(\xi)) = \frac{\delta(\xi-\xi_0)}{|f'(\xi_0)|}$$

which is valid if $f(\xi)$ is a monotonic function of ξ vanishing for $\xi = \xi_0$ [13]. Setting $\xi = t-ax$ in $(3)_3$, one finds that

$$\tau_{zz}^P(x, y, 0, t) = -a\delta(t-ax)H(y)H(x)$$

where the superscript P has been added to denote parallel motion. It immediately follows that

$$\tau_{zz}^{Q}(x, y, 0, t) = \lim_{c\to\infty} [c\tau_{zz}^{P}(x, y, 0, t)] = -\delta(t)H(y)H(x)$$

the superscript Q identifies the quantities pertaining to a load on one quadrant of the surface $z = 0$. By these arguments it follows that

$$g^{Q}(x, y, z, t) = \lim_{c\to\infty} [cg^{P}(x, y, z, t)]$$

for all the corresponding quantities. For example, in the indicated limit the last term of (32) gives the contribution

$$\delta_{jz}H(x)H(y)(a_1/v)\delta(t-a_{1z})$$

which corresponds to the plane wave directly under the first quadrant of the x–y plane found in [1]. The last term of (95) gives no contribution. As shown in [1], the solution for a uniform load acting on the rectangular region at $z = 0$ bounded by the lines $x = \pm a$ and $y = \pm b$ is given by

$$\begin{aligned} v_j^{R}(x, y, z, t) &= v_j^{Q}(x+a, y+b, z, t) - v_j^{Q}(x-a, y+b, z, t) \\ &\quad + v_j^{Q}(x-a, y-b, z, t) - v_j^{Q}(x+a, y-b, z, t) \end{aligned}$$

where the superscript R identifies the response when the plane is loaded on a finite rectangular region of its surface.

SUMMARY AND CONCLUSIONS

In the foregoing sections, exact expressions were derived for the transient response of a half-space to a normal finite load moving on its surface. The solution was found to be a superposition of cylindrical, hemispherical, conical and plane waves. It was found that the solution for a normal infinite moving line is contained in the solution given here. In fact, from equations (31) and (69), one can immediately write

$$\begin{aligned} \tilde{v}_j^{I}(x, y, z, p) &= (1-\delta_{jy})\frac{a}{\mu 2\pi i}\int_{-i\infty}^{i\infty} (s-a)^{-1}F_{jd}(is, 0, 1)e^{-p[\eta_d(is,0,1)z+xs]}\,ds \\ &\quad + (1-\delta_{jy})\frac{a}{\mu 2\pi i}\int_{-i\infty}^{i\infty} (s-a)^{-1}F_{js}(is, 0, 1)e^{-p[\eta_s(is,0,1)z+xs]}\,ds \end{aligned}$$

where the superscript I identifies the solution for a moving infinite line load [2]. From these results, one concludes that the solution is a superposition of that for an infinite line load, plus correction terms to account for the end points.

REFERENCES

[1] F. R. NORWOOD, Exact transient response of an elastic half-space loaded over a rectangular region of its surface. *J. appl. Mech.* **36**, 516–522 (1969).

[2] R. G. PAYTON, *Int. J. Engng Sci.* **5**, 49 (1967).

[3] D. D. ANG, *Q. appl. Math.* **18**, 251 (1960).

[4] R. G. PAYTON, *Int. J. Engng Sci.* **4**, 287 (1968).

[5] J. MIKLOWITZ, Elastic wave propagation, in *Applied Mechanics Surveys*. Spartan Books (1966).
[6] D. C. GAKENHEIMER, Transient excitation of an elastic half-space by a point load travelling on the surface, Ph.D. Thesis, California Institute of Technology (1968). Also *J. appl. Mech.* **36**, 505–515 (1969).
[7] Private communication. In a note to appear in *J. appl. Mech.* Gakenheimer solves the stationary line load problem by a limiting process.
[8] B. VAN DER POL and H. BREMMER, *Operational Calculus Based on the Two-Sided Laplace Transform*, 2nd edition. Cambridge University Press (1964).
[9] A. T. DE HOOP, *Appl. scient. Res.* **8**, Section B, 349 (1959).
[10] H. S. CARSLAW and J. C. JAEGER, *Operational Calculus in Applied Mathematics*. Oxford University Press (1941).
[11] M. MITRA, *Proc. Camb. phil. Soc. math. phys. Sci.* **60**, 683 (1964).
[12] E. T. COPSON, *An Introduction to the Theory of Functions of a Complex Variable*. Oxford University Press (1960).
[13] B. FRIEDMAN, *Principles and Techniques of Applied Mathematics*. John Wiley (1956).

(*Received* 7 *July* 1969; *revised* 12 *February* 1970)

Абстракт—Рассматривается воздействие упругого полупространства под влиянием импульсивной полубесконечной линейной нагрузки, движущейся параллельно к ее начальному положению. Найденное решение является наложением трех волн, а именно цилиндрической, сферической и конусной. Решение получено методом Канярда и путем обобщения действительного преобразования де Гупа на двойные интеграля фурье с сингулярностями на действительной оси переменных преобразования. Даются скорости волн внутри полупространства для обыкновенных значений отношения Пуассона, используя интегралы и альгебраические выражения. Получается случай для стационарной нагрузки, действующей на одном квадранте ограниченной поверхности. Путем простого наложения получается решение для полупространства нагруженного конечной линейной нагрузкой или загруженною в конечном прямоугольном районе.